

# On $q$ -analogues of quadratic Euler sums

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**Abstract** In this paper we define the generalized  $q$ -analogues of Euler sums and present a new family of identities for  $q$ -analogues of Euler sums by using the method of Jackson  $q$ -integral representations of series. We then apply it to obtain a family of identities relating quadratic Euler sums to linear sums and  $q$ -polylogarithms. Furthermore, we also use certain stuffle products to evaluate several  $q$ -series with  $q$ -harmonic numbers. Some interesting new results and illustrative examples are considered. Finally, we can obtain some explicit relations for the classical Euler sums when  $q$  approaches to 1.

**Keywords**  $q$ -harmonic number;  $q$ -Euler sum;  $q$ -polylogarithm function.

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## 1 Introduction

For positive integers  $m$  and  $k$ , let  $H_m^{(k)}$  and  $\overline{H}_m^{(k)}$  stand for the  $m$ -th generalized harmonic number and the  $m$ -th generalized alternating harmonic number defined by [2, 9, 22]

$$H_m^{(k)} := \sum_{j=1}^m \frac{1}{j^k}, \quad \overline{H}_m^{(k)} := \sum_{j=1}^m \frac{(-1)^{j-1}}{j^k},$$

respectively. If  $k > 1$ , the generalized harmonic number  $H_m^{(k)}$  converges to the (Riemann) zeta value  $\zeta(k)$ :

$$\lim_{m \rightarrow \infty} H_m^{(k)} = \zeta(k).$$

When  $k = 1$ ,  $H_m^{(1)} \equiv H_m$  (resp.  $\overline{H}_m^{(1)} \equiv \overline{H}_m$ ) is the classical harmonic number (resp. the classical alternating harmonic number).

Let  $n$  be a positive integer. Let  $k_1, \dots, k_n$  be nonzero integers and let  $k$  be a positive integer with  $k \geq 2$ . The classical Euler sums are defined by the convergent series

$$S(k_1, \dots, k_n; k) := \sum_{m=1}^{\infty} \frac{X_m(k_1) \cdots X_m(k_n)}{m^k},$$

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$$\overline{S}(k_1, \dots, k_n; k) := \sum_{m=1}^{\infty} \frac{X_m(k_1) \cdots X_m(k_n)}{m^k} (-1)^{m-1},$$

where

$$X_m(k) := \begin{cases} H_m^{(k)} & \text{if } k \geq 1, \\ \overline{H}_m^{(-k)} & \text{if } k \leq -1. \end{cases}$$

Here we call  $|k_1| + \cdots + |k_n| + k$  the weight, and  $n$  the depth. Throughout the paper, for a positive integer  $k$ , we use  $\bar{k}$  to denote the negative entry  $-k$ . For example, we have

$$S(1, \bar{2}, 3; 4) = S(1, -2, 3; 4), \quad \overline{S}(\bar{1}, \bar{2}, 3; 4) = \overline{S}(-1, -2, 3; 4).$$

A good deal of work on Euler sums has been focused on the problem of determining when complicated sums can be expressed in terms of simpler sums. Thus, researchers are interested in determining which sums can be expressed in terms of other sums of lesser depth. The origin of the study of Euler sums went back to the correspondence of Euler with Goldbach in 1742-1743 (see [12]) and Euler's paper [8] that appeared in 1776. Euler studied linear (or double) Euler sums and established some important formulas for them. For example, he proved that (see [4, 9])

$$S(1; k) = \frac{1}{2} \left\{ (k+2)\zeta(k+1) - \sum_{i=1}^{k-2} \zeta(k-i)\zeta(i+1) \right\}.$$

Moreover, Euler proved that the linear sums  $S(l; k)$  ( $l \geq 1, k \geq 2$ ) are reducible to zeta values whenever  $k+l$  is less than 7 or when  $k+l$  is odd and less than 13. Furthermore, he conjectured that the linear sums  $S(l; k)$  would be reducible to zeta values whenever  $k+l$  is odd, and even proposed the general formula. In [6], D. Borwein, J. M. Borwein and R. Girgensohn proved the conjecture, and in [4], D.H. Bailey, J. M. Borwein and R. Girgensohn demonstrated that it is "very likely" that the linear sums  $S(l; k)$  with  $k+l > 7$  and  $k+l$  even, are not reducible. After that many different methods, including partial fraction expansions, Eulerian Beta integrals, summation formulas for generalized hypergeometric functions and contour integrals, have been used to evaluate these sums (see [4, 6, 9]). For example, P. Flajolet and B. Salvy informed us about some ongoing work of theirs ([9]) to evaluate Euler sums in an entirely different way, namely using contour integration and the residue theorem. In this way they manage to prove, for example, that the sums  $S(1, 1, 1; k)$  with  $k = 2, 3, 4, 6$  can be evaluated in terms of zeta values. There are also a lot of recent contributions on nonlinear Euler sums (depth  $\geq 2$ ), see [21, 22]. For example, in [21], we proved that all Euler sums of the form  $S(k_1, k_2; k)$  with weight 4, 5, 6, 7, 9 are expressible polynomially in terms of zeta values. For weight 8, all such sums are the sum of a polynomial in zeta values and a rational multiple of  $S(2; 6)$ . And all weight 10 quadratic sums  $S(1, l; k)$  are reducible to  $S(2; 6)$  and  $S(2; 8)$ .

So far, surprising little work has been done on  $q$ -analogues of Euler sums. Actually, there are many possible ways to  $q$ -extend the Euler sums. Here we recall one  $q$ -analogue. Let  $q$  be a fixed real number with  $0 < q < 1$ . Let  $n$  be a positive integer. For a sequence  $\mathbf{k} = (k_1, \dots, k_n)$  of positive integers, a sequence  $\mathbf{x} = (x_1, \dots, x_n)$  of variables with  $-1 \leq x_i \leq 1$ , a positive integer  $k$  and a variable  $x$  with  $-1 < x < 1$ , we set

$$S \left[ \begin{matrix} \mathbf{k} \\ \mathbf{x} \end{matrix} \middle| x \right] \equiv S \left[ \begin{matrix} k_1, \dots, k_n \\ x_1, \dots, x_n \end{matrix} \middle| x \right] := \sum_{m=1}^{\infty} \frac{\zeta_m[k_1, x_1] \cdots \zeta_m[k_n, x_n]}{[m]^k} x^m, \quad (1.1)$$

where  $[m]$  denotes the  $q$ -analogue of a nonnegative integer, defined by

$$[m] \equiv [m]_q := \frac{1 - q^m}{1 - q},$$

and  $\zeta_m[k, x]$  is the partial sum of the  $q$ -polylogarithm function  $\text{Li}_k[x]$ , defined as

$$\zeta_m[k, x] := \sum_{j=1}^m \frac{x^j}{[j]^k}.$$

Here the  $q$ -polylogarithm function  $\text{Li}_k[x]$  is defined by

$$\text{Li}_k[x] := \sum_{m=1}^{\infty} \frac{x^m}{[m]^k}, \quad (-1 < x < 1).$$

Note that

$$\ln[1 - x] = -\text{Li}_1[x]$$

is the  $q$ -analogues of natural logarithm function. If  $n = 0$  in (1.1), we set

$$S \left[ \begin{matrix} \emptyset \\ \emptyset \end{matrix} \middle| \begin{matrix} k \\ x \end{matrix} \right] := \text{Li}_k[x].$$

When taking the limit  $q \rightarrow 1$  and  $x \rightarrow 1$  with  $x_j = 1$  in (1.1) we get

$$\lim_{q \rightarrow 1} S \left[ \begin{matrix} k_1, \dots, k_n \\ 1, \dots, 1 \end{matrix} \middle| \begin{matrix} k \\ 1 \end{matrix} \right] = S(k_1, \dots, k_n; k).$$

For a real number  $a$ , we set

$$H_k[x, a] := \sum_{m=1}^{\infty} \frac{x^{m+a}}{[m+a]^k},$$

where for a general real number  $b$ ,

$$[b] \equiv [b]_q := \frac{1 - q^b}{1 - q}.$$

There are fewer results for sums of the type (1.1). Some related results for  $q$ -Euler type sums may be seen in the works of [7, 15, 17–20, 23, 24] and references therein. The second author jointly with M. Zhang and W. Zhu [23] proved that for positive integer  $k \geq 2$ , the linear sum

$$S \left[ \begin{matrix} 1 \\ 1 \end{matrix} \middle| \begin{matrix} k \\ q \end{matrix} \right]$$

can be expressed as a rational linear combination of products of  $q$ -polylogarithms, the quadratic sum

$$S \left[ \begin{matrix} 1, 1 \\ 1, 1 \end{matrix} \middle| \begin{matrix} k \\ q \end{matrix} \right]$$

and the cubic combination sum

$$S \left[ \begin{matrix} 1, 1, 1 \\ 1, 1, 1 \end{matrix} \middle| \begin{matrix} k \\ q \end{matrix} \right] - 3S \left[ \begin{matrix} 1, 2 \\ 1, 1 \end{matrix} \middle| \begin{matrix} k \\ q \end{matrix} \right]$$

are reducible to linear  $q$ -sums and to polynomials in  $q$ -polylogarithms. Some simple examples are

$$\begin{aligned} S \left[ \begin{matrix} 1 \\ 1 \end{matrix} \middle| \begin{matrix} 2 \\ q \end{matrix} \right] &= \text{Li}_3 [q] + \text{Li}_3 [q^2], \\ S \left[ \begin{matrix} 1 \\ 1 \end{matrix} \middle| \begin{matrix} 3 \\ q \end{matrix} \right] &= \frac{3}{2} \text{Li}_4 [q^2] + \text{Li}_4 [q] - \frac{1}{2} \text{Li}_2^2 [q], \\ S \left[ \begin{matrix} 1, 1 \\ 1, 1 \end{matrix} \middle| \begin{matrix} 2 \\ q \end{matrix} \right] &= \frac{7}{2} \text{Li}_4 [q^2] + 2 \text{Li}_4 [q] - \frac{1}{2} \text{Li}_2^2 [q] - (1-q) (\text{Li}_3 [q^2] + \text{Li}_3 [q]). \end{aligned}$$

We continue the study of  $q$ -Euler sums in this paper. The purpose of the paper is to prove the following theorems.

**Theorem 1.1.** *Let  $k, l$  be positive integers and  $a, b, x$  be real numbers with  $a, b \neq -1, -2, \dots$  and  $|x| < 1$ . Then the following identity holds:*

$$\begin{aligned} &(-1)^{k-1} \sum_{m=1}^{\infty} \frac{q^{(m+b)k}}{[m+b]^{k+l}} \sum_{j=1}^m \frac{x^{j+a+b}}{[j+a+b]} - (-1)^{l-1} \sum_{m=1}^{\infty} \frac{q^{(m+a)l}}{[m+a]^{k+l}} \sum_{j=1}^m \frac{x^{j+a+b}}{[j+a+b]} \\ &= \sum_{j=1}^{l-1} (-1)^{j-1} H_{k+j}[q^{j-1}x, a] H_{l+1-j}[x, b] - \sum_{j=1}^{k-1} (-1)^{j-1} H_{l+j}[q^{j-1}x, b] H_{k+1-j}[x, a] \\ &\quad + (-1)^{l-1} \left( H_1[x, b] H_{k+l}[q^{l-1}x, a] - H_1[x, a+b] H_{k+l}[q^l, a] \right) \\ &\quad - (-1)^{k-1} \left( H_1[x, a] H_{k+l}[q^{k-1}x, b] - H_1[x, a+b] H_{k+l}[q^k, b] \right). \end{aligned} \tag{1.2}$$

**Theorem 1.2.** *Let  $k, l$  be positive integers and  $s, h, x$  be real numbers with  $p > s \geq 0, m > h \geq 0$  and  $|x| < 1$ . Then we have*

$$\begin{aligned} &(-1)^{k-1} S \left[ \begin{matrix} l, 1 \\ q^s, q^h x \end{matrix} \middle| \begin{matrix} k \\ q^{k-h} \end{matrix} \right] - (-1)^{l-1} S \left[ \begin{matrix} k, 1 \\ q^h, q^s x \end{matrix} \middle| \begin{matrix} l \\ q^{l-s} \end{matrix} \right] \\ &= \sum_{j=1}^{l-1} (-1)^{j-1} \text{Li}_{l+1-j}[q^s x] S \left[ \begin{matrix} k \\ q^h \end{matrix} \middle| \begin{matrix} j \\ q^{j-1} x \end{matrix} \right] - \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^h x] S \left[ \begin{matrix} l \\ q^s \end{matrix} \middle| \begin{matrix} j \\ q^{j-1} x \end{matrix} \right] \\ &\quad + (-1)^{l-1} \ln[1 - q^s x] \left( S \left[ \begin{matrix} k \\ q^h \end{matrix} \middle| \begin{matrix} l \\ q^{l-s} \end{matrix} \right] - S \left[ \begin{matrix} k \\ q^h \end{matrix} \middle| \begin{matrix} l \\ q^{l-1} x \end{matrix} \right] \right) \\ &\quad - (-1)^{k-1} \ln[1 - q^h x] \left( S \left[ \begin{matrix} l \\ q^s \end{matrix} \middle| \begin{matrix} k \\ q^{k-h} \end{matrix} \right] - S \left[ \begin{matrix} l \\ q^s \end{matrix} \middle| \begin{matrix} k \\ q^{k-1} x \end{matrix} \right] \right). \end{aligned} \tag{1.3}$$

**Theorem 1.3.** *For positive integers  $k$  and  $l$ , it holds*

$$\begin{aligned} &(-1)^{k-1} S \left[ \begin{matrix} l+1, 1 \\ q^l, q \end{matrix} \middle| \begin{matrix} k \\ q \end{matrix} \right] + (-1)^{l-1} S \left[ \begin{matrix} k, 1 \\ q^{k-1}, q \end{matrix} \middle| \begin{matrix} l+1 \\ q \end{matrix} \right] \\ &= \text{Li}_{l+1}[q^l] \text{Li}_{k+1}[q^k] + \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^{k-j}] S \left[ \begin{matrix} j \\ q \end{matrix} \middle| \begin{matrix} l+1 \\ q^l \end{matrix} \right] \\ &\quad + (-1)^{k-1} \text{Li}_{l+1}[q^l] S \left[ \begin{matrix} 1 \\ q \end{matrix} \middle| \begin{matrix} k \\ q \end{matrix} \right] - \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^{k-j}] \text{Li}_{l+j+1}[q^{l+1}] \end{aligned}$$

$$- \sum_{j=1}^{l-1} (-1)^{j-1} \text{Li}_{l+1-j}[q^{l-j}] S \left[ \begin{matrix} k \\ q^{k-1} \end{matrix} \middle| \begin{matrix} j+1 \\ q \end{matrix} \right]. \quad (1.4)$$

**Theorem 1.4.** *Let  $n, k_1, \dots, k_n$  be positive integers and  $x_1, \dots, x_n$  be real numbers with  $|x_j| < 1$ . we have*

$$\prod_{j=1}^n \text{Li}_{k_j}[x_j] = \sum_{j=0}^{n-1} \sum_{1 \leq i_1 < \dots < i_j \leq n} (-1)^{n-1-j} S \left[ \begin{matrix} k_{i_1}, \dots, k_{i_j} \\ x_{i_1}, \dots, x_{i_j} \end{matrix} \middle| \frac{(k_1 + \dots + k_n) - (k_{i_1} + \dots + k_{i_j})}{(x_1 \dots x_n)/(x_{i_1} \dots x_{i_j})} \right].$$

We prove Theorems 1.1-1.3 in Section 2 by calculating the Jackson  $q$ -integral of  $q$ -polylogarithm functions, and prove Theorem 1.4 in Section 3 algebraically. In the last section we give some interesting identities (known or new) involving harmonic numbers.

## 2 Proofs of Theorems 1.1, 1.2 and 1.3

We prove Theorems 1.1, 1.2 and 1.3 in this section by calculating the Jackson  $q$ -integral of  $q$ -polylogarithm functions.

### 2.1 Jackson $q$ -integral

The Jackson  $q$ -integral and  $q$ -derivative are defined by ([1, 3, 5, 13, 23])

$$\int_a^x f(t) d_q t := (1-q) \sum_{i=0}^{\infty} q^i [x f(q^i x) - a f(q^i a)],$$

$$D_q f(x) := \frac{f(qx) - f(x)}{qx - x},$$

respectively. For example, we have

$$D_q(H_k[x, a]) = \frac{H_{k-1}[x, a]}{x}, \quad D_q(x^m) = [m]x^{m-1}.$$

And it is easy to verify that

$$D_q(f(x)g(x)) = g(qx)D_q(f(x)) + f(x)D_q(g(x)) = f(qx)D_q(g(x)) + g(x)D_q(f(x)),$$

$$D_q \left( \int_a^x f(t) d_q t \right) = f(x), \quad \int_a^x D_q(f(t)) d_q t = f(x) - f(a),$$

$$\int_a^x f(t) D_q(g(t)) d_q t = [f(t)g(t)]_a^x - \int_a^x g(qt) D_q(f(t)) d_q t.$$

### 2.2 Proof of Theorem 1.1

To prove Theorem 1.1, we need a lemma.

**Lemma 2.1.** Let  $m, k$  be positive integers and  $a, b, x$  be real numbers with  $a, b \neq -1, -2, \dots$  and  $|x| < 1$ . Then the following identity holds:

$$\begin{aligned} \int_0^x H_k[t, a] t^{m+b-1} d_q t &= \sum_{j=1}^{k-1} (-1)^{j-1} \frac{q^{(m+b)(j-1)} x^{m+b}}{[m+b]^j} H_{k+1-j}[x, a] \\ &\quad + (-1)^{k-1} \frac{q^{(m+b)(k-1)}}{[m+b]^k} \left( x^{m+b} H_1[x, a] - q^{m+b} H_1[x, a+b] \right) \\ &\quad + (-1)^{k-1} \frac{q^{(m+b)k}}{[m+b]^k} \sum_{j=1}^m \frac{x^{j+a+b}}{[j+a+b]}. \end{aligned} \quad (2.1)$$

**Proof.** Denote the left hand-side of (2.1) by  $I_k$ . Then we have

$$I_k = \frac{1}{[m+b]} \int_0^x H_k[t, a] D_q(t^{m+b}) d_q t = \frac{x^{m+b}}{[m+b]} H_k[x, a] - \frac{q^{m+b}}{[m+b]} I_{k-1},$$

and

$$\begin{aligned} I_1 &= \frac{x^{m+b}}{[m+b]} H_1[x, a] - \frac{q^{m+b}}{[m+b]} \int_0^x \frac{t^{m+a+b}}{1-t} d_q t \\ &= \frac{x^{m+b}}{[m+b]} H_1[x, a] + \frac{q^{m+b}}{[m+b]} \sum_{j=1}^m \frac{x^{j+a+b}}{[j+a+b]} - \frac{q^{m+b}}{[m+b]} H_1[x, a+b]. \end{aligned}$$

Hence we get (2.1) by induction on  $k$ . □

**Proof of Theorem 1.1.** Considering the Jackson  $q$ -integral

$$\begin{aligned} \int_0^x \frac{H_k[t, a] H_l[t, b]}{t} d_q t &= \sum_{m=1}^{\infty} \frac{1}{[m+a]^k} \int_0^x H_l[t, b] t^{m+a-1} d_q t \\ &= \sum_{m=1}^{\infty} \frac{1}{[m+b]^l} \int_0^x H_k[t, a] t^{m+b-1} d_q t, \end{aligned}$$

we get (1.2) with the help of (2.1). □

Setting  $x = q$  and  $k = l = 1$  in Theorem 1.1, we get

$$\begin{aligned} &\sum_{m=1}^{\infty} \frac{q^{m+b}}{[m+b]^2} \sum_{j=1}^m \frac{q^{j+a+b}}{[j+a+b]} - \sum_{m=1}^{\infty} \frac{q^{m+a}}{[m+a]^2} \sum_{j=1}^m \frac{q^{j+a+b}}{[j+a+b]} \\ &= [a] H_2[q, a] \sum_{m=1}^{\infty} \frac{q^{m+b}}{[m+b][m+a+b]} - [b] H_2[q, b] \sum_{m=1}^{\infty} \frac{q^{m+a}}{[m+a][m+a+b]}. \end{aligned}$$

### 2.3 Proof of Theorem 1.2

Similarly as the proof of Theorem 1.1, we give a proof of Theorem 1.2.

**Proof of Theorem 1.2.** Using (2.1), we compute the Jackson  $q$ -integral

$$\begin{aligned}
& \int_0^x \frac{\text{Li}_l[q^s t] \text{Li}_k[q^h t]}{t(1-t)} d_q t = \sum_{m=1}^{\infty} \zeta_m[l, q^s] \int_0^x t^{m-1} \text{Li}_k[q^h t] d_q t \\
&= \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^h x] \sum_{m=1}^{\infty} \frac{\zeta_m[l, q^s]}{[m]^j} (q^{j-1} x)^m \\
&\quad + (-1)^{k-1} \ln[1 - q^h x] \sum_{m=1}^{\infty} \frac{\zeta_m[l, q^s]}{[m]^k} (q^{(k-h)m} - (q^{k-1} x)^m) \\
&\quad + (-1)^{k-1} \sum_{m=1}^{\infty} \frac{\zeta_m[l, q^s] \zeta_m[1, q^h x]}{[m]^k} q^{(k-h)m} \\
&= \sum_{j=1}^{l-1} (-1)^{j-1} \text{Li}_{l+1-j}[q^s x] \sum_{m=1}^{\infty} \frac{\zeta_m[k, q^h]}{[m]^j} (q^{j-1} x)^m \\
&\quad + (-1)^{l-1} \ln[1 - q^s x] \sum_{m=1}^{\infty} \frac{\zeta_m[k, q^h]}{[m]^l} (q^{(l-s)m} - (q^{l-1} x)^m) \\
&\quad + (-1)^{l-1} \sum_{m=1}^{\infty} \frac{\zeta_m[k, q^h] \zeta_m[1, q^s x]}{[m]^l} q^{(l-s)m},
\end{aligned}$$

from which we get (1.3). □

Setting  $x \rightarrow 1$  in Theorem 1.2, we obtain

$$\begin{aligned}
& (-1)^{k-1} S \left[ \begin{matrix} l, 1 \\ q^s, q^h \end{matrix} \middle| \begin{matrix} k \\ q^{k-h} \end{matrix} \right] - (-1)^{l-1} S \left[ \begin{matrix} k, 1 \\ q^h, q^s \end{matrix} \middle| \begin{matrix} l \\ q^{l-s} \end{matrix} \right] \\
&= \sum_{j=2}^{l-1} (-1)^{j-1} \text{Li}_{l+1-j}[q^s] S \left[ \begin{matrix} k \\ q^h \end{matrix} \middle| \begin{matrix} j \\ q^{j-1} \end{matrix} \right] - \sum_{j=2}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^h] S \left[ \begin{matrix} l \\ q^s \end{matrix} \middle| \begin{matrix} j \\ q^{j-1} \end{matrix} \right] \\
&\quad + (-1)^{l-1} \ln[1 - q^s] \left( S \left[ \begin{matrix} k \\ q^h \end{matrix} \middle| \begin{matrix} l \\ q^{l-s} \end{matrix} \right] - S \left[ \begin{matrix} k \\ q^h \end{matrix} \middle| \begin{matrix} l \\ q^{l-1} \end{matrix} \right] \right) \\
&\quad - (-1)^{k-1} \ln[1 - q^h] \left( S \left[ \begin{matrix} l \\ q^s \end{matrix} \middle| \begin{matrix} k \\ q^{k-h} \end{matrix} \right] - S \left[ \begin{matrix} l \\ q^s \end{matrix} \middle| \begin{matrix} k \\ q^{k-1} \end{matrix} \right] \right) \\
&\quad + \sum_{m=1}^{\infty} \frac{\text{Li}_l[q^s] \zeta_m[k, q^h] - \text{Li}_k[q^h] \zeta_m[l, q^s]}{[m]}. \tag{2.2}
\end{aligned}$$

To evaluate the last sum in the right-hand side of (2.2), we use

**Theorem 2.2.** Let  $k, l$  be positive integers and  $x, y, z$  be real numbers with  $|x|, |y|, |z| < 1$ . Then we have

$$\begin{aligned}
& \sum_{m=1}^{\infty} \frac{\text{Li}_k[x] \zeta_m[l, y] - \text{Li}_l[y] \zeta_m[k, x]}{[m]} z^m \\
&= \text{Li}_l[y] S \left[ \begin{matrix} 1 \\ z \end{matrix} \middle| \begin{matrix} k \\ x \end{matrix} \right] - \text{Li}_k[x] S \left[ \begin{matrix} 1 \\ z \end{matrix} \middle| \begin{matrix} l \\ y \end{matrix} \right] + \text{Li}_k[x] \text{Li}_{l+1}[zy] - \text{Li}_l[y] \text{Li}_{k+1}[zx]. \tag{2.3}
\end{aligned}$$

Taking  $x = q^h, y = q^s$  and  $z \rightarrow 1$  in (2.3), and using (2.2), we have

**Corollary 2.3.** *Let  $s, h$  be positive reals and  $k, l$  be positive integers with  $l > \max\{s, 1\}$  and  $k > \max\{h, 1\}$ . Then it holds*

$$\begin{aligned}
& (-1)^{k-1} S \left[ \begin{matrix} l, 1 \\ q^s, q^h \end{matrix} \middle| \begin{matrix} k \\ q^{k-h} \end{matrix} \right] - (-1)^{l-1} S \left[ \begin{matrix} k, 1 \\ q^h, q^s \end{matrix} \middle| \begin{matrix} l \\ q^{l-s} \end{matrix} \right] \\
&= \sum_{j=2}^{l-1} (-1)^{j-1} \text{Li}_{l+1-j}[q^s] S \left[ \begin{matrix} k \\ q^h \end{matrix} \middle| \begin{matrix} j \\ q^{j-1} \end{matrix} \right] - \sum_{j=2}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^h] S \left[ \begin{matrix} l \\ q^s \end{matrix} \middle| \begin{matrix} j \\ q^{j-1} \end{matrix} \right] \\
&\quad + (-1)^{l-1} \ln[1 - q^s] \left( S \left[ \begin{matrix} k \\ q^h \end{matrix} \middle| \begin{matrix} l \\ q^{l-s} \end{matrix} \right] - S \left[ \begin{matrix} k \\ q^h \end{matrix} \middle| \begin{matrix} l \\ q^{l-1} \end{matrix} \right] \right) \\
&\quad - (-1)^{k-1} \ln[1 - q^h] \left( S \left[ \begin{matrix} l \\ q^s \end{matrix} \middle| \begin{matrix} k \\ q^{k-h} \end{matrix} \right] - S \left[ \begin{matrix} l \\ q^s \end{matrix} \middle| \begin{matrix} k \\ q^{k-1} \end{matrix} \right] \right) \\
&\quad + \text{Li}_k[q^h] S \left[ \begin{matrix} 1 \\ 1 \end{matrix} \middle| \begin{matrix} l \\ q^s \end{matrix} \right] - \text{Li}_l[q^s] S \left[ \begin{matrix} 1 \\ 1 \end{matrix} \middle| \begin{matrix} k \\ q^h \end{matrix} \right] + \text{Li}_l[q^s] \text{Li}_{k+1}[q^h] - \text{Li}_k[q^h] \text{Li}_{l+1}[q^s].
\end{aligned}$$

Finally, we give a proof of Theorem 2.2.

**Proof of Theorem 2.2.** We compute the  $N$ -th partial sum of the series of the left-hand side of (2.3)

$$\begin{aligned}
& \sum_{m=1}^N \frac{\text{Li}_k[x] \zeta_m[l, y] - \text{Li}_l[y] \zeta_m[k, x]}{[m]} z^m \\
&= \text{Li}_k[x] \sum_{m=1}^N \frac{\zeta_m[l, y]}{[m]} z^m - \text{Li}_l[y] \sum_{m=1}^N \frac{\zeta_m[k, x]}{[m]} z^m \\
&= \text{Li}_k[x] \sum_{m=1}^N \sum_{j=1}^m \frac{y^j z^m}{[m][j]^l} - \text{Li}_l[y] \sum_{m=1}^N \sum_{j=1}^m \frac{x^j z^m}{[m][j]^k} \\
&= \text{Li}_k[x] \sum_{j=1}^N \sum_{m=j}^N \frac{y^j z^m}{[m][j]^l} - \text{Li}_l[y] \sum_{j=1}^N \sum_{m=j}^N \frac{x^j z^m}{[m][j]^k} \\
&= \text{Li}_k[x] \sum_{j=1}^N \frac{\zeta_N[1, z] - \zeta_{j-1}[1, z]}{[j]^l} y^j - \text{Li}_l[y] \sum_{j=1}^N \frac{\zeta_N[1, z] - \zeta_{j-1}[1, z]}{[j]^k} x^j \\
&= \zeta_N[1, z] (\text{Li}_k[x] \zeta_N[l, y] - \text{Li}_l[y] \zeta_N[k, x]) + \text{Li}_l[y] \sum_{j=1}^N \frac{\zeta_{j-1}[1, z]}{[j]^k} x^j - \text{Li}_k[x] \sum_{j=1}^N \frac{\zeta_{j-1}[1, z]}{[j]^l} y^j.
\end{aligned}$$

Letting  $N$  tend to infinity, we get (2.3).  $\square$

## 2.4 Proof of Theorem 1.3

To prove Theorem 1.3, we need the following lemmas.

**Lemma 2.4.** *For positive integers  $k$  and  $i$ , it holds*

$$\sum_{m=1}^{\infty} \frac{\zeta_m[k, q^{k-1}]}{[m][m+i]} q^m = \frac{1}{[i]} \left\{ \begin{aligned} & \text{Li}_{k+1}[q^k] + (-1)^{k-1} \sum_{j=1}^{i-1} \frac{[H_j^{(1)}]}{[j]^k} q^j \\ & + \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^{k-j}] [H_{i-1}^{(j)}] \end{aligned} \right\}, \quad (2.4)$$



where we set [23]

$$\left[ H_m^{(k)} \right] = \zeta_m[k, q].$$

**Proof.** By using the Cauchy product of power series and the definition of  $q$ -harmonic numbers, we have

$$\sum_{m=1}^{\infty} \zeta_m[k, q^l] x^m = \frac{\text{Li}_k[q^l x]}{1-x},$$

where  $l$  is any positive integer and  $x$  is any real number with  $|x| < 1$ . Multiplied by  $x^{-1} - x^{i-1}$  and  $q$ -integrated over  $(0, q)$ , the above equation yields

$$[i] \sum_{m=1}^{\infty} \frac{\zeta_m[k, q^l]}{[m][m+i]} q^m = \text{Li}_{k+1}[q^{l+1}] + \sum_{j=1}^{i-1} \sum_{m=1}^{\infty} \frac{q^{(l+1)m+j}}{[m]^k [m+j]}.$$

Taking  $l = k - 1$  in above equation, we obtain

$$[i] \sum_{m=1}^{\infty} \frac{\zeta_m[k, q^{k-1}]}{[m][m+i]} q^m = \text{Li}_{k+1}[q^k] + \sum_{j=1}^{i-1} q^j \sum_{m=1}^{\infty} \frac{q^{km}}{[m]^k [m+j]},$$

which together with the formula

$$\sum_{m=1}^{\infty} \frac{q^{km}}{[m]^k [m+j]} = \sum_{p=1}^{k-1} \frac{(-1)^{p-1}}{[j]^p} \text{Li}_{k-p+1}[q^{k-p}] + (-1)^{k-1} \frac{[H_j^{(1)}]}{[j]^k} \quad (2.5)$$

yield the desired result.  $\square$

**Lemma 2.5.** For any positive integers  $k_1, k_2$  and any real numbers  $x, y$  with  $|x|, |y| < 1$ , we have

$$S \left[ \begin{matrix} k_1 & k_2 \\ x & y \end{matrix} \right] + S \left[ \begin{matrix} k_2 & k_1 \\ y & x \end{matrix} \right] = \text{Li}_{k_1}[x] \text{Li}_{k_2}[y] + \text{Li}_{k_1+k_2}[xy]. \quad (2.6)$$

**Proof.** We consider the generating function

$$F_2[x, y, z] := \sum_{m=1}^{\infty} (\zeta_m[k_1, x] \zeta_m[k_2, y] - \zeta_m[k_1 + k_2, xy]) z^{m-1},$$

where  $|z| < 1$ . By the definition of  $\zeta_m[k, x]$ , we have

$$\begin{aligned} F_2[x, y, z] &= \sum_{m=1}^{\infty} \left\{ \left( \zeta_m[k_1, x] + \frac{x^{m+1}}{[m+1]^{k_1}} \right) \left( \zeta_m[k_2, y] + \frac{y^{m+1}}{[m+1]^{k_2}} \right) \right. \\ &\quad \left. - \left( \zeta_m[k_1 + k_2, xy] + \frac{x^{m+1} y^{m+1}}{[m+1]^{k_1+k_2}} \right) \right\} z^m \\ &= z F_2[x, y, z] + \sum_{m=1}^{\infty} \left( \frac{\zeta_m[k_1, x]}{[m+1]^{k_2}} y^{m+1} + \frac{\zeta_m[k_2, y]}{[m+1]^{k_1}} x^{m+1} \right) z^m \\ &= z F_2[x, y, z] + \sum_{m=1}^{\infty} \left( \frac{\zeta_m[k_1, x]}{[m]^{k_2}} y^m + \frac{\zeta_m[k_2, y]}{[m]^{k_1}} x^m - 2 \frac{x^m y^m}{[m]^{k_1+k_2}} \right) z^{m-1}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} F_2[x, y, z] &= \sum_{m=1}^{\infty} \left( \frac{\zeta_m[k_1, x]}{[m]^{k_2}} y^m + \frac{\zeta_m[k_2, y]}{[m]^{k_1}} x^m - 2 \frac{x^m y^m}{[m]^{k_1+k_2}} \right) \frac{z^{m-1}}{1-z} \\ &= \sum_{m=1}^{\infty} \sum_{j=1}^m \left( \frac{\zeta_j[k_1, x]}{[j]^{k_2}} y^j + \frac{\zeta_j[k_2, y]}{[j]^{k_1}} x^j - 2 \frac{x^j y^j}{[j]^{k_1+k_2}} \right) z^{m-1}. \end{aligned}$$

Then equating coefficients of  $z^{m-1}$ , we establish the relation

$$\sum_{j=1}^m \left( \frac{\zeta_j[k_1, x]}{[j]^{k_2}} y^j + \frac{\zeta_j[k_2, y]}{[j]^{k_1}} x^j \right) = \zeta_m[k_1, x] \zeta_m[k_2, y] + \zeta_m[k_1 + k_2, xy].$$

Letting  $m$  tend to infinity in above equation, we deduce (2.6).  $\square$

**Remark 2.1.** Similarly, considering the following function

$$F_3[x, y, z, t] := \sum_{m=1}^{\infty} (\zeta_m[k_1, x] \zeta_m[k_2, y] \zeta_m[k_3, z] - \zeta_m[k_1 + k_2 + k_3, xyz]) t^{m-1},$$

and applying the same arguments as in the proof of (2.6), we may deduce the following formula

$$\begin{aligned} &S \left[ \begin{matrix} k_1, k_2 \\ x, y \end{matrix} \middle| \begin{matrix} k_3 \\ z \end{matrix} \right] + S \left[ \begin{matrix} k_1, k_3 \\ x, z \end{matrix} \middle| \begin{matrix} k_2 \\ y \end{matrix} \right] + S \left[ \begin{matrix} k_2, k_3 \\ y, z \end{matrix} \middle| \begin{matrix} k_1 \\ x \end{matrix} \right] \\ &= S \left[ \begin{matrix} k_1 \\ x \end{matrix} \middle| \begin{matrix} k_2 + k_3 \\ yz \end{matrix} \right] + S \left[ \begin{matrix} k_2 \\ y \end{matrix} \middle| \begin{matrix} k_1 + k_3 \\ xz \end{matrix} \right] + S \left[ \begin{matrix} k_3 \\ z \end{matrix} \middle| \begin{matrix} k_1 + k_2 \\ xy \end{matrix} \right] \\ &\quad + \text{Li}_{k_1}[x] \text{Li}_{k_2}[y] \text{Li}_{k_3}[z] - \text{Li}_{k_1+k_2+k_3}[xyz], \end{aligned} \quad (2.7)$$

where  $k_1, k_2, k_3$  are positive integers and  $x, y, z$  are real number with  $|x|, |y|, |z| < 1$ . It is clear that (2.6) and (2.7) are immediate corollaries of Theorem 1.4.

**Lemma 2.6.** For any positive integers  $k_1, k_2$  and any real numbers  $x, y, z$  with  $|x|, |y|, |z| < 1$ , we have

$$\sum_{m=1}^{\infty} \frac{x^m}{[m]^{k_1}} \sum_{j=1}^m \frac{z^j}{[j]^{k_2}} \zeta_j[1, y] + S \left[ \begin{matrix} k_1, 1 \\ x, y \end{matrix} \middle| \begin{matrix} k_2 \\ z \end{matrix} \right] = \text{Li}_{k_1}[x] S \left[ \begin{matrix} 1 \\ y \end{matrix} \middle| \begin{matrix} k_2 \\ z \end{matrix} \right] + S \left[ \begin{matrix} 1 \\ y \end{matrix} \middle| \begin{matrix} k_1 + k_2 \\ xz \end{matrix} \right]. \quad (2.8)$$

**Proof.** Replacing  $y$  by  $zt$  in (2.6), then dividing it by  $\frac{1}{1-t}$  and  $q$ -integrating over the interval  $(0, y)$ , we can deduce the desired result.  $\square$

Finally, we come to the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Set

$$\sum = \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} \frac{\zeta_m[k, q^{k-1}]}{[m][m+i][i]^l} q^{m+li}.$$

On the one hand, using (2.4), we have

$$\sum = \sum_{i=1}^{\infty} \frac{q^{li}}{[i]^{l+1}} \left\{ \begin{aligned} &\text{Li}_{k+1}[q^k] + (-1)^{k-1} \sum_{j=1}^{i-1} \frac{[H_j^{(1)}]}{[j]^k} q^j \\ &+ \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^{k-j}] [H_{i-1}^{(j)}] \end{aligned} \right\}$$

$$\begin{aligned}
&= (-1)^{k-1} \sum_{m=1}^{\infty} \frac{q^{lm}}{[m]^{l+1}} \sum_{j=1}^m \frac{[H_j^{(1)}]}{[j]^k} q^j - (-1)^{k-1} S \left[ \begin{matrix} 1 \\ q \end{matrix} \middle| \begin{matrix} k+l+1 \\ q^{l+1} \end{matrix} \right] \\
&\quad + \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^{k-j}] S \left[ \begin{matrix} j \\ q \end{matrix} \middle| \begin{matrix} l+1 \\ q^l \end{matrix} \right] + \text{Li}_{l+1}[q^l] \text{Li}_{k+1}[q^k] \\
&\quad - \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^{k-j}] \text{Li}_{l+j+1}[q^{l+1}].
\end{aligned}$$

Setting  $k_1 = l+1, k_2 = k, x = q^l, y = z = q$  in (2.8), we get

$$\sum_{m=1}^{\infty} \frac{q^{lm}}{[m]^{l+1}} \sum_{j=1}^m \frac{q^j}{[j]^k} [H_j^{(1)}] - S \left[ \begin{matrix} 1 \\ q \end{matrix} \middle| \begin{matrix} k+l+1 \\ q^{l+1} \end{matrix} \right] = \text{Li}_{l+1}[q^l] S \left[ \begin{matrix} 1 \\ q \end{matrix} \middle| \begin{matrix} k \\ q \end{matrix} \right] - S \left[ \begin{matrix} l+1, 1 \\ q^l, q \end{matrix} \middle| \begin{matrix} k \\ q \end{matrix} \right],$$

which deduce that

$$\begin{aligned}
\sum &= (-1)^{k-1} \text{Li}_{l+1}[q^l] S \left[ \begin{matrix} 1 \\ q \end{matrix} \middle| \begin{matrix} k \\ q \end{matrix} \right] - (-1)^{k-1} S \left[ \begin{matrix} l+1, 1 \\ q^l, q \end{matrix} \middle| \begin{matrix} k \\ q \end{matrix} \right] \\
&\quad + \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^{k-j}] S \left[ \begin{matrix} j \\ q \end{matrix} \middle| \begin{matrix} l+1 \\ q^l \end{matrix} \right] + \text{Li}_{l+1}[q^l] \text{Li}_{k+1}[q^k] \\
&\quad - \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^{k-j}] \text{Li}_{l+j+1}[q^{l+1}].
\end{aligned} \tag{2.9}$$

On the other hand, using (2.5), we get

$$\begin{aligned}
\sum &= \sum_{m=1}^{\infty} \frac{\zeta_m[k, q^{k-1}]}{[m]} q^m \sum_{i=1}^{\infty} \frac{q^{li}}{[i]^l [m+i]} \\
&= \sum_{m=1}^{\infty} \frac{\zeta_m[k, q^{k-1}]}{[m]} q^m \left( \sum_{j=1}^{l-1} \frac{(-1)^{j-1}}{[m]^j} \text{Li}_{l-j+1}[q^{l-j}] + (-1)^{l-1} \frac{[H_m^{(1)}]}{[m]^l} \right) \\
&= \sum_{j=1}^{l-1} (-1)^{j-1} \text{Li}_{l+1-j}[q^{l-j}] S \left[ \begin{matrix} k \\ q^{k-1} \end{matrix} \middle| \begin{matrix} j+1 \\ q \end{matrix} \right] + (-1)^{l-1} S \left[ \begin{matrix} k, 1 \\ q^{k-1}, q \end{matrix} \middle| \begin{matrix} l+1 \\ q \end{matrix} \right].
\end{aligned} \tag{2.10}$$

Comparing (2.9) and (2.10), we get the result.  $\square$

### 3 Proof of Theorem 1.4

In this section, we use the stuffle product to give a proof of Theorem 1.4.

For a sequence  $\mathbf{k} = (k_1, \dots, k_n)$  of positive integers, the weight and the depth of  $\mathbf{k}$  are defined by

$$\text{wt}(\mathbf{k}) = k_1 + \dots + k_n, \quad \text{dep}(\mathbf{k}) = n,$$

respectively. For an empty sequence, we set  $\text{wt}(\emptyset) = \text{dep}(\emptyset) = 0$ . We call  $\mathbf{l}$  a subsequence of  $\mathbf{k}$ , if there exist integers  $m, i_1, \dots, i_m$  with  $0 \leq m \leq n$  and  $1 \leq i_1 < \dots < i_m \leq n$ , such

that  $\mathbf{l} = (k_{i_1}, \dots, k_{i_m})$ . Let  $\text{Sub}(\mathbf{k})$  be the set of all subsequences of  $\mathbf{k}$ . If  $\mathbf{x} = (x_1, \dots, x_n)$  is a sequence of variables, we set  $|\mathbf{x}| = x_1 \cdots x_n$ . And for any  $\mathbf{l} = (k_{i_1}, \dots, k_{i_m}) \in \text{Sub}(\mathbf{k})$ , we set

$$\mathbf{x}_{\mathbf{l}} = (x_{i_1}, \dots, x_{i_m}).$$

Note that  $|\emptyset| = 1$ ,  $\mathbf{x}_{\emptyset} = \emptyset$  and  $\mathbf{x}_{\mathbf{k}} = \mathbf{x}$ . Therefore Theorem 1.4 may be rewritten as

**Theorem 3.1.** *Let  $n$  be a positive integer,  $\mathbf{k} = (k_1, \dots, k_n)$  be a sequence of positive integers and  $\mathbf{x} = (x_1, \dots, x_n)$  be a sequence of real numbers with  $|x_j| < 1$ . we have*

$$\prod_{j=1}^n \text{Li}_{k_j}[x_j] = \sum_{\mathbf{l} \in \text{Sub}(\mathbf{k}), \mathbf{l} \neq \mathbf{k}} (-1)^{n - \text{dep}(\mathbf{l}) - 1} S \left[ \begin{matrix} \mathbf{l} \\ \mathbf{x}_{\mathbf{l}} \end{matrix} \middle| \frac{\text{wt}(\mathbf{k}) - \text{wt}(\mathbf{l})}{|\mathbf{x}|/|\mathbf{x}_{\mathbf{l}}|} \right]. \quad (3.1)$$

To prove Theorem 3.1, we use the stuffle product. Similar as in [10, 11], let

$$\mathcal{M} := \left\{ \begin{bmatrix} k \\ x \end{bmatrix} \middle| (k, x) \in \mathbb{N} \times (-1, 1) \right\},$$

which we regard as an alphabet with noncommutative letters. Let  $\mathcal{M}^*$  be the set of all words generated by  $\mathcal{M}$ , which contains the empty word  $1_{\mathcal{M}}$ . We denote a nonempty word  $\begin{bmatrix} k_1 \\ x_1 \end{bmatrix} \cdots \begin{bmatrix} k_n \\ x_n \end{bmatrix}$  simplify by  $\begin{bmatrix} k_1, \dots, k_n \\ x_1, \dots, x_n \end{bmatrix}$ . Let  $\mathfrak{h}^1 = \mathbb{Q}\langle \mathcal{M} \rangle$  be the noncommutative polynomial algebra over  $\mathbb{Q}$  generated by  $\mathcal{M}$ . As a rational vector space,  $\mathfrak{h}^1$  has a basis  $\mathcal{M}^*$ .

We now define the stuffle product  $\bar{*}$  on the algebra  $\mathfrak{h}^1$ , which is  $\mathbb{Q}$ -bilinear, and satisfies the following axioms

- (1)  $1_{\mathcal{M}} \bar{*} w = w \bar{*} 1_{\mathcal{M}} = w$  for any  $w \in \mathcal{M}^*$ ;
- (2)  $au \bar{*} bv = a(u \bar{*} bv) + b(au \bar{*} v) - (a \circ b)(u \bar{*} v)$  for any  $a, b \in \mathcal{M}$  and any  $u, v \in \mathcal{M}^*$ .

Here we set

$$\begin{bmatrix} k \\ x \end{bmatrix} \circ \begin{bmatrix} l \\ y \end{bmatrix} := \begin{bmatrix} k+l \\ xy \end{bmatrix}.$$

Then by [11, 16], the product  $\bar{*}$  is commutative and associative.

For any  $w \in \mathfrak{h}^1$ , we define a function  $\text{Li}^*[w]$  by  $\mathbb{Q}$ -linearity,  $\text{Li}^*[1_{\mathcal{M}}] = 1$  and

$$\text{Li}^* \begin{bmatrix} k_1, \dots, k_n \\ x_1, \dots, x_n \end{bmatrix} := \sum_{m_1 \geq \dots \geq m_n \geq 1} \frac{x_1^{m_1} \cdots x_n^{m_n}}{[m_1]^{k_1} \cdots [m_n]^{k_n}}.$$

Then we have

$$\text{Li}^* \begin{bmatrix} k \\ x \end{bmatrix} = \text{Li}_k[x].$$

Immediately from the definitions, we have

**Lemma 3.2.** (1) *For any  $w_1, w_2 \in \mathfrak{h}^1$ , we have*

$$\text{Li}^*[w_1 \bar{*} w_2] = \text{Li}^*[w_1] \text{Li}^*[w_2].$$

(2) Let  $n$  be a positive integer and  $w_1 = \begin{bmatrix} k_1 \\ x_1 \end{bmatrix}, \dots, w_n = \begin{bmatrix} k_n \\ x_n \end{bmatrix}, w = \begin{bmatrix} k \\ x \end{bmatrix} \in \mathcal{M}$ . Then we have

$$S \left[ \begin{matrix} k_1, \dots, k_n \\ x_1, \dots, x_n \end{matrix} \middle| \begin{matrix} k \\ x \end{matrix} \right] = \text{Li}^\star[w(w_1 \bar{*} \dots \bar{*} w_n)].$$

**Proof.** One can prove (1) similarly as in [11, 16], and prove (2) similar as in [14].  $\square$

We prove the corresponding equation of (3.1) in the algebra  $\mathfrak{h}^1$ .

**Theorem 3.3.** Let  $n$  be a positive integer and  $w_1 = \begin{bmatrix} k_1 \\ x_1 \end{bmatrix}, \dots, w_n = \begin{bmatrix} k_n \\ x_n \end{bmatrix} \in \mathcal{M}$ . Set  $\mathbf{k} = (k_1, \dots, k_n)$  and  $\mathbf{x} = (x_1, \dots, x_n)$ , then we have

$$w_1 \bar{*} \dots \bar{*} w_n = \sum_{\substack{\mathbf{l} = (k_{i_1}, \dots, k_{i_m}) \in \text{Sub}(\mathbf{k}) \\ \mathbf{l} \neq \mathbf{k}}} (-1)^{n-m-1} \left[ \frac{\text{wt}(\mathbf{k}) - \text{wt}(\mathbf{l})}{|\mathbf{x}|/|\mathbf{x}_\mathbf{l}|} \right] (w_{i_1} \bar{*} \dots \bar{*} w_{i_m}). \quad (3.2)$$

**Proof.** We proceed on induction on  $n$ . The case of  $n = 1$  is trivial. Now assume that (3.2) is proved for  $\mathbf{k}$  and  $\mathbf{x}$ . For any  $w_{n+1} = \begin{bmatrix} k_{n+1} \\ x_{n+1} \end{bmatrix} \in \mathcal{M}$ , set  $\mathbf{k}' = (k_1, \dots, k_n, k_{n+1})$  and  $\mathbf{x}' = (x_1, \dots, x_n, x_{n+1})$ . Using the induction assumption and the definition of the stuffle product, we have

$$\begin{aligned} & w_1 \bar{*} \dots \bar{*} w_n \bar{*} w_{n+1} \\ &= \sum_{\substack{\mathbf{l} = (k_{i_1}, \dots, k_{i_m}) \in \text{Sub}(\mathbf{k}) \\ \mathbf{l} \neq \mathbf{k}}} (-1)^{n-m-1} w_{n+1} \bar{*} \left( \left[ \frac{\text{wt}(\mathbf{k}) - \text{wt}(\mathbf{l})}{|\mathbf{x}|/|\mathbf{x}_\mathbf{l}|} \right] (w_{i_1} \bar{*} \dots \bar{*} w_{i_m}) \right) \\ &= \sum_{\substack{\mathbf{l} = (k_{i_1}, \dots, k_{i_m}) \in \text{Sub}(\mathbf{k}) \\ \mathbf{l} \neq \mathbf{k}}} (-1)^{n-m-1} w_{n+1} \left( \left[ \frac{\text{wt}(\mathbf{k}) - \text{wt}(\mathbf{l})}{|\mathbf{x}|/|\mathbf{x}_\mathbf{l}|} \right] (w_{i_1} \bar{*} \dots \bar{*} w_{i_m}) \right) \\ &\quad + \sum_{\substack{\mathbf{l} = (k_{i_1}, \dots, k_{i_m}) \in \text{Sub}(\mathbf{k}) \\ \mathbf{l} \neq \mathbf{k}}} (-1)^{n-m-1} \left[ \frac{\text{wt}(\mathbf{k}) - \text{wt}(\mathbf{l})}{|\mathbf{x}|/|\mathbf{x}_\mathbf{l}|} \right] (w_{n+1} \bar{*} w_{i_1} \bar{*} \dots \bar{*} w_{i_m}) \\ &\quad + \sum_{\substack{\mathbf{l} = (k_{i_1}, \dots, k_{i_m}) \in \text{Sub}(\mathbf{k}) \\ \mathbf{l} \neq \mathbf{k}}} (-1)^{n-m} \left[ \frac{\text{wt}(\mathbf{k}) - \text{wt}(\mathbf{l}) + k_{n+1}}{|\mathbf{x}|x/|\mathbf{x}_\mathbf{l}|} \right] (w_{i_1} \bar{*} \dots \bar{*} w_{i_m}) \\ &= w_{n+1} (w_1 \bar{*} \dots \bar{*} w_n) \\ &\quad + \sum_{\substack{\mathbf{l} = (k_{i_1}, \dots, k_{i_m}) \in \text{Sub}(\mathbf{k}) \\ \mathbf{l} \neq \mathbf{k}}} (-1)^{n-m-1} \left[ \frac{\text{wt}(\mathbf{k}) - \text{wt}(\mathbf{l})}{|\mathbf{x}|/|\mathbf{x}_\mathbf{l}|} \right] (w_{n+1} \bar{*} w_{i_1} \bar{*} \dots \bar{*} w_{i_m}) \\ &\quad + \sum_{\substack{\mathbf{l} = (k_{i_1}, \dots, k_{i_m}) \in \text{Sub}(\mathbf{k}) \\ \mathbf{l} \neq \mathbf{k}}} (-1)^{n-m} \left[ \frac{\text{wt}(\mathbf{k}) - \text{wt}(\mathbf{l}) + k_{n+1}}{|\mathbf{x}|x/|\mathbf{x}_\mathbf{l}|} \right] (w_{i_1} \bar{*} \dots \bar{*} w_{i_m}). \end{aligned}$$

Since any  $\mathbf{l} \in \text{Sub}(\mathbf{k}')$  with  $\mathbf{l} \neq \mathbf{k}'$  must satisfy and only satisfy one of the following conditions

- (i)  $\mathbf{l} = (k_1, \dots, k_n)$ ;
- (ii)  $\mathbf{l} = (k_{i_1}, \dots, k_{i_m}, k_{n+1})$  with  $(k_{i_1}, \dots, k_{i_m}) \neq \mathbf{k}$ ;

(iii)  $1 = (k_{i_1}, \dots, k_{i_m}) \neq \mathbf{k}$  and  $i_m < n + 1$ ,

we prove (3.2) for  $\mathbf{k}'$  and  $\mathbf{x}'$ . □

Finally, we can prove Theorem 3.1.

**Pooof of Theorem 3.1.** Applying  $\text{Li}^*$  on both sides of (3.2), and with the helps of Lemma 3.2, we get the result. □

## 4 Some identities on Euler sums

From Theorems 1.1-1.3, taking  $x \rightarrow \pm 1, q \rightarrow 1$ , we get the following corollaries.

**Corollary 4.1.** *For positive integers  $k > 1$  and  $l > 1$ , it holds*

$$\begin{aligned} & (-1)^{k-1}S(1, l; k) - (-1)^{l-1}S(1, k; l) \\ &= \sum_{j=2}^{l-1} (-1)^{j-1} \zeta(l+1-j)S(k; j) - \sum_{j=2}^{k-1} (-1)^{j-1} \zeta(k+1-j)S(l; j) \\ & \quad + \zeta(k)S(1; l) - \zeta(l)S(1; k) + \zeta(l)\zeta(k+1) - \zeta(k)\zeta(l+1). \end{aligned}$$

**Corollary 4.2.** *For positive integers  $k > 1$  and  $l$ , it holds*

$$\begin{aligned} & (-1)^{l-1}S(1, k; l+1) + (-1)^{k-1}S(1, l+1; k) \\ &= \zeta(k+1)\zeta(l+1) + \sum_{j=1}^{k-1} (-1)^{j-1} \zeta(k+1-j)S(j; l+1) + (-1)^{k-1} \zeta(l+1)S(1; k) \\ & \quad - \sum_{j=1}^{k-1} (-1)^{j-1} \zeta(k+1-j)\zeta(l+j+1) - \sum_{j=1}^{l-1} (-1)^{j-1} \zeta(l+1-j)S(k; j+1). \end{aligned}$$

**Corollary 4.3.** *Let  $l \geq 2$  and  $k \geq 0$  be integers. Then we have*

$$\begin{aligned} & (-1)^l [S(\bar{1}, l+2k+1; l) + S(\bar{1}, l; l+2k+1)] \\ &= \sum_{j=1}^{l+2k} (-1)^{j-1} \bar{\zeta}(l+2k+2-j)S(l; \bar{j}) \\ & \quad - \sum_{j=1}^{l-1} (-1)^{j-1} \bar{\zeta}(l+1-j)S(l+2k+1; \bar{j}) \\ & \quad + (-1)^l \ln 2 [S(l+2k+1; l) + S(l; l+2k+1)] \\ & \quad + (-1)^l \ln 2 [S(l+2k+1; \bar{l}) + S(l; \overline{l+2k+1})]. \end{aligned}$$

**Corollary 4.4.** *For integers  $l \in \mathbb{N} \setminus \{1\}$  and  $k \in \mathbb{N} \cup \{0\}$ , we have*

$$\begin{aligned} & (-1)^l [S(\bar{1}, l+2k; l) - S(\bar{1}, l; l+2k)] \\ &= \sum_{j=1}^{l+2k-1} (-1)^{j-1} \bar{\zeta}(l+2k+1-j)S(l; \bar{j}) \\ & \quad - \sum_{j=1}^{l-1} (-1)^{j-1} \bar{\zeta}(l+1-j)S(l+2k; \bar{j}) \end{aligned}$$

$$\begin{aligned}
& + (-1)^l \ln 2 [S(l+2k; l) - S(l; l+2k)] \\
& + (-1)^l \ln 2 [S(l+2k; \bar{l}) - S(l; \overline{l+2k})].
\end{aligned}$$

From Theorem 1.4, we find for a positive integer  $l > 1$ , it holds

$$\begin{aligned}
\zeta^4(l) &= 4S(\{l\}_3; l) - 6S(\{l\}_2; 2l) + 4S(l; 3l) - \zeta(4l), \\
\zeta(2l)\zeta^2(l) &= 2S(l, 2l; l) + S(\{l\}_2; 2l) - S(2l; 2l) - 2S(l; 3l) + \zeta(4l), \\
\zeta(3l)\zeta^2(l) &= 2S(l, 3l; l) + S(\{l\}_2; 3l) - S(3l; 2l) - 2S(l; 4l) + \zeta(5l), \\
\zeta^5(l) &= 5S(\{l\}_4; l) - 10S(\{l\}_3; 2l) + 10S(\{l\}_2; 3l) - 5S(l; 4l) + \zeta(5l), \\
\zeta(2l)\zeta^3(l) &= S(\{l\}_3; 2l) + 3S(\{l\}_2, 2l; l) - 3S(\{l\}_2; 3l) - 3S(l, 2l; 2l) \\
& + 3S(l; 4l) + S(2l; 3l) - \zeta(5l).
\end{aligned}$$

Here  $\{l\}_d$  denotes the sequence  $\underbrace{l, \dots, l}_{d \text{ times}}$ .

**Corollary 4.5.** *For integers  $l \in \mathbb{N} \setminus \{1\}$  and  $k \in \mathbb{N} \cup \{0\}$ , then the following identity holds:*

$$\begin{aligned}
& S(\bar{1}, l+2k+1; l) + S(\bar{1}, l; l+2k+1) + S(l, l+2k+1; \bar{1}) \\
& = S(l; \overline{l+2k+2}) + S(\bar{1}; 2l+2k+1) + S(l+2k+1; \overline{l+1}) \\
& + \ln 2 \zeta(l+2k+1) \zeta(l) - \bar{\zeta}(2l+2k+2).
\end{aligned}$$

Hence, from Corollary 4.3 and Corollary 4.5, we obtain the following description of quadratic Euler sums.

**Corollary 4.6.** *For  $l \in \mathbb{N} \setminus \{1\}$  and  $k \in \mathbb{N} \cup \{0\}$ , the alternating quadratic sums*

$$S(l, l+2k+1; \bar{1}) = \sum_{n=1}^{\infty} \frac{H_n^{(l)} H_n^{(l+2k+1)}}{n} (-1)^{n-1}$$

*are reducible to linear sums.*

A simple example is as follows:

$$\begin{aligned}
S(2, 3; \bar{1}) &= -\frac{161}{64} \zeta(6) + \frac{31}{16} \zeta(5) \ln 2 + \frac{9}{32} \zeta^2(3) + \frac{3}{8} \zeta(2) \zeta(3) \ln 2 + 2\zeta(2) \text{Li}_4\left(\frac{1}{2}\right) \\
& - \frac{5}{4} \zeta(4) \ln^2 2 + \frac{1}{12} \zeta(2) \ln^4 2 + S(2; \bar{4}) - S(\bar{3}; 3).
\end{aligned}$$

In fact, proceeding in a similar fashion to evaluation of the Theorem 1.2 and Corollary 4.6, it is possible to evaluate other Euler sums involving harmonic numbers and alternating harmonic numbers. For example, in the same way as in the proof of Corollary 4.6, we also prove that the alternating quadratic sums

$$S(\bar{l}, \overline{l+2k+1}; \bar{1}) = \sum_{n=1}^{\infty} \frac{\overline{H}_n^{(l)} \overline{H}_n^{(l+2k+1)}}{n} (-1)^{n-1}$$

are reducible to linear sums, for  $l \in \mathbb{N} \setminus \{1\}$  and  $k \in \mathbb{N} \cup \{0\}$ . A special case is as follows:

$$S(\bar{2}, \bar{3}; \bar{1}) = \frac{163}{128} \zeta(6) - \frac{31}{16} \zeta(5) \ln 2 + \frac{3}{16} \zeta^2(3) - \frac{3}{4} \zeta(2) \zeta(3) \ln 2 - \zeta(2) \text{Li}_4\left(\frac{1}{2}\right)$$

$$+ \frac{5}{8}\zeta(4)\ln^2 2 - \frac{1}{24}\zeta(2)\ln^4 2 + S(\bar{2}; 4) + S(\bar{3}; 3).$$

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